EXACTLY SOLVED LATTICE MODELS AND THE STATISTICAL PHYSICS OF POLYMER CHAINS

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1. Introduction

The concept of integrability has some magic in that certain complicated many-body problems can be solved exactly. The great majority of these problems are model systems in two-dimensional statistical mechanics. Such exactly solved lattice models have played an important role in our understanding of phase transitions and critical phenomena.[1] There have also been rather surprising connections with formerly diverse physical and mathematical areas such as conformal field theory, quantum groups, topology and knot theory.

Here I give a brief account of results obtained for the statistical physics of two-dimensional polymer chains from the viewpoint of exactly solved lattice models.

The configurational behaviour of long flexible self-avoiding polymer chains, both in the bulk and near a surface, has long been of interest.[2–4] The relevant lattice model is the O(n) model, which describes self-avoiding walks in the $n \to 0$ limit.[2–4] We particularly consider the O(n) loop model on the honeycomb lattice,[5,6] which turns out to be exactly solvable.

2. Walks in the Bulk

The modelling of self-avoiding polymer chain configurations has long been of difficulty because of their memory. Although certain walk problems can be treated which are locally self-avoiding, they nevertheless become globally random.[7] Genuinely self-avoiding configurations can be formulated in terms of a transfer matrix eigenvalue problem via the O(n) model. This formalism can be shown to work successfully for various counting or enumeration problems.[8,9] The transfer matrix eigenspectrum has been obtained exactly by means of the Bethe ansatz for a number of different boundary conditions. Baxter, whose interest lay in a related colouring problem on the triangular lattice, solved the model under periodic boundary conditions.[10] Then using some remarkable relations which connect the operator content, or critical exponents, of two-dimensional lattice models to the finite-size behaviour of the transfer matrix eigenspectrum,[11] we were able to recover the familiar Nienhuis results[6] for self-avoiding walks in the bulk.[12–14] Exact results have also been obtained for a more general O(n) loop model on the square lattice.[15–17] Although this model does not strictly describe self-avoiding walks at $n = 0$, it reduces to the honeycomb model in a certain limit.[15]

The number of L-step self-avoiding walks from some origin deep in the bulk scales as

$$Z \sim \mu^L L^{\gamma^{\mathrm{b}}-1} \quad \text{where} \quad \gamma^{\mathrm{b}} = \tfrac{43}{32} \quad \text{and} \quad \mu = \sqrt{2+\sqrt{2}}.$$

The configurational exponent γ^{b} is *universal* in that it holds for other planar lattices. However, the connective constant μ is lattice-dependent. Its exact value is known only for the honeycomb lattice.

3. Walks Near a Surface

The study of self-avoiding walks near a surface began in earnest with the work of Barber et al.[18] From the exactly solved model viewpoint, one is interested in *open* boundary conditions as a means of investigating such surface critical phenomena. Yet the investigation of exactly solved lattice models with open boundaries has lagged behind the investigation of exactly solved lattice models in the bulk, i.e. with periodic boundary conditions. However, this situation is rapidly changing. Our own contributions in this area have been motivated by the related polymer problems, for which we have obtained new exact results by solving the $\mathrm{O}(n)$ loop model with *open* boundary conditions.[19–22] In particular, an exact solution has been found for two sets of boundary weights.[19,20] In the language of surface critical phenomena,[23] these results precisely correspond to the $\mathrm{O}(n)$ model at the *ordinary* and *special* surface transitions. For the boundary weights corresponding to the special transition, exact results follow in the limit $n \to 0$ for the *adsorption* of a flexible self-avoiding polymer chain at the boundary.

Physically, there is an adsorbed phase for $T < T_a$ and a desorbed phase for $T > T_a$, where T_a is the critical adsorption temperature. Again for the honeycomb lattice, we find the exact result

$$\mathrm{e}^{\epsilon/kT_a} = \sqrt{1+\sqrt{2}} = 1.553\ldots,$$

where ϵ is the monomer energy for surface contact. The universal critical exponents,

$$\gamma_1^{\mathrm{o}} = \tfrac{61}{64} \quad \text{and} \quad \gamma_1^{\mathrm{sp}} = \tfrac{93}{64},$$

which again follow from the finite-size corrections to the transfer matrix eigenspectra, govern the number of configurations of walks with one end attached to the boundary:

$$Z_1 \sim \begin{cases} \mu^L L^{\gamma_1^{\mathrm{o}}-1} & T > T_a, \\ \mu^L L^{\gamma_1^{\mathrm{sp}}-1} & T = T_a. \end{cases}$$

A further configurational exponent,

$$\gamma_1 = \tfrac{85}{64},$$

is obtained by considering *mixed* ordinary and special boundary conditions.[24,25] This exponent controls the number of self-avoiding configurations beginning from near some origin at the surface with a non-adsorbing boundary to the left and an adsorbing boundary to the right (see Fig. 1). The same adsorption temperature T_a holds for the mixed boundary case. These results have recently been numerically confirmed in the half-plane using series expansion techniques.[26]

Although we have obtained various exponents by explicit calculations on the honeycomb lattice, they are expected to be valid for other planar lattices on *universality* grounds.

4. Walks on the Manhattan Lattice

Another class of self-avoiding walks that can be solved in the same sense corresponds to a different $\mathrm{O}(n)$ loop model on the square lattice.[27,28] It turns out that at different values in the parameter space this model describes both interacting self-avoiding walks on the Manhattan lattice at the collapse temperature and Hamiltonian walks on the Manhattan lattice, allowing a derivation of their relevant bulk and surface properties.[29,30] The interacting self-avoiding walk model has essentially the same critical behaviour as the Duplantier-Saleur model for θ-point polymers in two dimensions.[31]

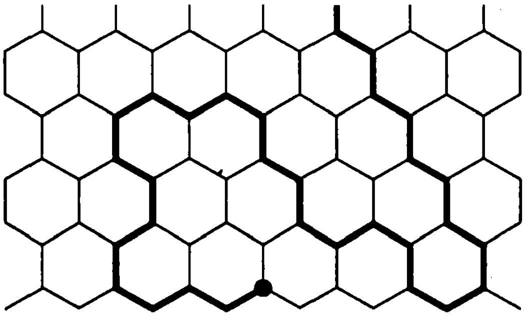

Figure 1: A self-avoiding walk from an origin on the surface of the half-plane.

5. Fully Packed Walks

Exact results can also be obtained for self-avoiding (Hamiltonian) walks which visit every site of the honeycomb lattice. The underlying lattice model was considered long ago by Baxter in the context of a colouring problem,[32] and later by others in the context of walks.[33–35] More generally it can be considered as a *fully packed* $O(n)$ loop model.[36,37] The exact results obtained include the value of the entropy, from which it follows that for self-avoiding walks on the honeycomb lattice, the entropy loss per step due to compactness, relative to the freedom of open configurations, is given by[37]

$$\frac{1}{2}\log\left[\frac{3\sqrt{3}}{4(2+\sqrt{2})}\right] = -0.483\,161\ldots$$

Acknowledgements

The work reported here has been part of an ongoing investigation supported by the Australian Research Council. It is a great pleasure to thank F. C. Alcaraz, M. N. Barber, R. J. Baxter, H. W. J. Blöte, J. L. Cardy, C. J. Hamer, B. I. Henry, A. Klümper, A. Kuniba, B. Nienhuis, A. L. Owczarek, K. A. Seaton, J. Suzuki, S. O. Warnaar and C. M. Yung for their collaboration and correspondence during this period.

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